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Convergence of Weighted Averages of Martingales in Banach Function Spaces

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Let $f = (f_n)_{n \geq 1}$ be a martingale and $(w_n)_{n \geq 1}$ a sequence of positive numbers such that $W_n = \sum_{k=1}^{n} w_k \to \infty$. Kazamaki and Tsuchikura proved that f converges in L^p $(1 < p < \infty)$ if and only if the weighted average $(\sigma_n(f))_{n \geq 1}$ of f converges in L^p, where $\sigma_n(f)$ are given by

$$\sigma_n(f) = \frac{1}{W_n} \sum_{k=1}^{n} w_k f_k, \qquad n = 1, 2, \dots .$$

We shall investigate the convergence of f and $\sigma_n(f)$ in general Banach function spaces X. Our main result can be applied to the case where X is a rearrangement-invariant space, or X is a weighted L^p-space with a weight function satisfying the condition A_p introduced by Izumisawa and Kazamaki.

Key Words: martingale, weighted average, Banach function space, rearrangement-invariant space.

INTRODUCTION

Let $(w_n)_{n \geq 1}$ be a sequence of positive numbers such that

$$W_n := \sum_{k=1}^{n} w_k \to \infty \quad \text{as } n \to \infty.$$

We shall call (w_n) a *weight sequence*. The weighted average $(\sigma_n(f))$ of a martingale $f = (f_n)_{n \geq 1}$ (with respect to (w_n)) is given by

$$\sigma_n(f) = \frac{1}{W_n} \sum_{k=1}^{n} w_k f_k, \qquad n = 1, 2, \dots . \tag{1}$$

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Kazamaki and Tsuchikura proved in [7] that $\sup_n \|\sigma_n(f)\|_p < \infty$ if and only if $\sup_n \|f_n\|_p < \infty$, where $1 \leq p < \infty$. From this fact and the martingale convergence theorem (see, e.g., [4, p. 28]), it follows that $f = (f_n)$ converges in L^p if and only if $(\sigma_n(f))$ converges in L^p for any fixed $1 < p < \infty$. We shall see later that the result is also valid for $p = 1$ or $p = \infty$ (cf. Theorem 2).

In this paper, we shall investigate the relationship between the convergence of $f = (f_n)$ and the convergence of $(\sigma_n(f))$ in general Banach function spaces X (see Definition 1). We want to characterize Banach function spaces X such that a martingale $f = (f_n)$ converges in X if and only if $(\sigma_n(f))$ converges in X for any weight sequence (w_n). According to Lemma 2 below, if (w_n) increases very rapidly, then $f = (f_n)$ and $(\sigma_n(f))$ converge simultaneously in any Banach function space X. As Example 1 shows, however, there exists a martingale $f = (f_n)$ such that $(\sigma_n(f))$ converges in X while f itself does not converge in X. We shall give a sufficient condition which assures that $f = (f_n)$ and $(\sigma_n(f))$ converge simultaneously in X for any weight sequence (w_n) (Theorem 2). In the case where X is a weighted L^p space, the condition is nearly necessary (cf. Theorem 4 and Example 1).

1. DEFINITIONS AND NOTATION

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. Unless otherwise stated, we shall consider martingales $f = (f_n)_{n \geq 1}$ on $(\Omega, \mathscr{F}, P)$; to specify the filtration relative to which $f = (f_n)$ is a martingale, we also write $f = (f_n, \mathscr{F}_n)_{n \geq 1}$.

DEFINITION 1. A Banach space $(X, \|\cdot\|_X)$ of (equivalence classes of) random variables on Ω is said to be a *Banach function space* if X has the following properties:

(a) $L^\infty \hookrightarrow X \hookrightarrow L^1$;

(b) if $|x| \leq |y|$ a.s. and $y \in X$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$;

(c) if $0 \leq x_n \uparrow x$ a.s., $x_n \in X$, and $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_n \|x_n\|_X$.

Let X be a Banach function space. We say that $x \in X$ has *absolutely continuous norm* if $\|x 1_{A_n}\|_X \downarrow 0$ whenever $A_n \in \mathscr{F}$ and $A_n \downarrow \varnothing$ a.s. If every $x \in X$ has absolutely continuous norm then X itself is said to have *absolutely continuous norm*. If X has absolutely continuous norm, $|x_n| \leq y \in X$ and $x_n \to x$ a.s., then $\|x_n - x\|_X \to 0$. Hence L^∞ is dense in X, if X has absolutely continuous norm. For details, see [1, pp. 14–16].

Many function spaces which come up in probability theory are rearrangement-invariant.

DEFINITION 2. A Banach function space X is said to be *rearrangement-invariant* (r.i.) if X has the following property: if x and y have the same distribution and $y \in X$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

For each $x \in L^1$, we denote by x^* the *nonincreasing rearrangement of* x given by

$$x^*(t) = \inf\{\lambda > 0: P(|x| > \lambda) \leq t\}, \qquad 0 \leq t \leq 1,$$

with the convention that $\inf \varnothing = \infty$. Note that x^* and $|x|$ have the same distribution, i.e.,

$$P(|x| > \lambda) = \mu(t \in [0,1]: x^*(t) > \lambda), \qquad \lambda \geq 0,$$

where μ stands for Lebesgue measure on the interval $[0,1]$ (cf. [1, p. 39]).

When the underlying probability space Ω contains an atom, it is sometimes useful to deal with universally rearrangement-invariant spaces rather than r.i. spaces.

DEFINITION 3. A Banach function space X is said to be *universally rearrangement-invariant* (u.r.i.) if X satisfies the following condition: if $\int_0^t x^*(s)\,ds \leq \int_0^t y^*(s)\,ds$ for every $t \in [0,1]$ and $y \in X$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$.

Clearly, every u.r.i. space X is r.i.; when Ω contains no atom, every r.i. space X is u.r.i. (cf. [1, p. 90; 10, p. 115]).

For example, L^p-spaces, Orlicz spaces L^Φ (cf. [1, p. 270]) and Lorentz spaces $L^{p,q}$ (cf. [1, p. 216]) are r.i. spaces. In fact, Luxemburg's representation theorem shows that these spaces are u.r.i.; a Banach function space X is u.r.i. if and only if there exists a r.i. space $\tilde{X}$ over the probability space $([0,1], \mu)$ such that

$$\|x\|_X = \|x^*\|_{\tilde{X}} \tag{2}$$

for every $x \in X$ (see [1, p. 90] or [10, p. 121]). Note that, if $X = L^\Phi(\Omega)$ and $\tilde{X} = L^\Phi([0,1])$, then Eq. (2) holds for every $x \in X$. Therefore, L^Φ is u.r.i., and in the same way, L^p and $L^{p,q}$ are u.r.i.

2. CONVERGENCE OF WEIGHTED AVERAGE OF MARTINGALES

In this section, we shall investigate the convergence and boundedness of the weighted average of martingales in a Banach function space X.

Let $1 \le p \le \infty$ and $\mathscr{G}$ be a sub-σ-field of $\mathscr{F}$. Then we have

$$\| E[x \mid \mathscr{G}] \|_p \le \|x\|_p \quad \text{for every } x \in L^p.$$

This fundamental inequality for conditional expectation is frequently used in the theory of martingales and means that $E[\cdot|\mathscr{G}]$ is a bounded linear operator on L^p into itself. Furthermore, if Φ is an increasing convex function with $\Phi(0) = 0$, then Jensen's inequality gives that, for every $x \in L^\Phi$,

$$E\left[\Phi\left(\frac{|E[x \mid \mathscr{G}]|}{\|x\|_\Phi}\right)\right] \le E\left[E\left[\Phi\left(\frac{|x|}{\|x\|_\Phi}\right)\middle|\mathscr{G}\right]\right]$$
$$= E\left[\Phi\left(\frac{|x|}{\|x\|_\Phi}\right)\right] \le 1.$$

This shows that $\|E[x \mid \mathscr{G}]\|_\Phi \le \|x\|_\Phi$ and hence $E[\cdot|\mathscr{G}]$ is a bounded linear operator of norm one on L^Φ into itself.

Let X be an arbitrary Banach function space. Then $E[\cdot|\mathscr{G}]$ may not be a bounded linear operator on X into itself (cf. [9]). In [8, 9], we proved that, when X has absolutely continuous norm, every uniformly integrable martingale $f = (f_n, \mathscr{F}_n)_{n \ge 1}$ converges in X if and only if each $E[\cdot|\mathscr{F}_n]$ is a bounded linear operator on X into itself and $\sup_n \|E[\cdot|\mathscr{F}_n]\| < \infty$. We shall see that this hypothesis on $E[\cdot|\mathscr{F}_n]$ is crucial for our present purpose also.

In what follows, we denote by $(\mathscr{L}(X), \|\cdot\|_{\mathscr{L}(X)})$ the space of bounded linear operators on X into itself, and by E_n the conditional expectation operator $E[\cdot|\mathscr{F}_n]$ for a given filtration $(\mathscr{F}_n)_{n \ge 1}$.

Theorem 1 and Corollary 1 of [7] are special cases of the following theorem.

THEOREM 1. *Let X be a Banach function space, $f = (f_n, \mathscr{F}_n)_{n \ge 1}$ a martingale, and $(w_n)_{n \ge 1}$ a weight sequence. If $E_n \in \mathscr{L}(X)$ for all $n \ge 1$ and $C := \sup_n \|E_n\|_{\mathscr{L}(X)} < \infty$, then*

$$\sup_n \| \sigma_n(f) \|_X \le \sup_n \|f_n\|_X \le C \sup_n \|\sigma_n(f)\|_X, \tag{3}$$

where $\sigma_n(f)$ is given by Eq. (1).

Proof. The left inequality is clear from the simple inequality

$$\|\sigma_n(f)\|_X \le \frac{1}{W_n} \sum_{k=1}^n w_k \|f_k\|_X.$$

To prove the right inequality, observe that for all $m \ge n$,

$$E[\sigma_m(f) \mid \mathscr{F}_n] = \frac{W_n}{W_m}\sigma_n(f) + \frac{W_m - W_n}{W_m} f_n. \tag{4}$$

This implies that for every $m > n$,

$$\frac{W_m - W_n}{W_m} \|f_n\|_X \leq \frac{W_n}{W_m} \| \sigma_n(f) \|_X + \| E[\sigma_m(f) \mid \mathscr{F}_n] \|_X$$

$$\leq \frac{W_n}{W_m} \sup_n \| \sigma_n(f) \|_X + C \sup_n \| \sigma_n(f) \|_X .$$

Letting $m \to \infty$, we obtain the right inequality of Eq. (3), since $W_m \to \infty$. ∎

COROLLARY 1. *Suppose that X is a u.r.i. space (or r.i. space if Ω contains no atom). Then a martingale $f = (f_n)_{n \geq 1}$ is bounded in X if and only if $(\sigma_n(f))_{n \geq 1}$ is bounded in X, and in that case*

$$\sup_n \|f_n\|_X = \sup_n \| \sigma_n(f) \|_X .$$

Proof. In view of Theorem 1, it suffices to show that $E_n \in \mathscr{L}(X)$ and $\|E_n\|_{\mathscr{L}(X)} \leq 1$ for each $n \geq 1$. This is an immediate consequence of Calderón's work [2]. We shall give here a more direct proof via the formula

$$\int_0^t x^*(s)\, ds = \inf\left\{ \|x_1\|_1 + t\|x_2\|_\infty : x = x_1 + x_2,\, x_1 \in L^1,\, x_2 \in L^\infty \right\},$$

which is valid for all $x \in L^1$ and $t \in [0,1]$. For the proof, see [1, p. 74]. Suppose that $x = x_1 + x_2$, $x_1 \in L^1$, and $x_2 \in L^\infty$. Then we have

$$\int_0^t (E_n x)^*(s)\, ds \leq \|E_n x_1\|_1 + t\|E_n x_2\|_\infty \leq \|x_1\|_1 + t\|x_2\|_\infty,$$

since E_n is an operator of norm one from L^1 (or L^∞) into itself. Taking the infimum of the right-hand side, we obtain

$$\int_0^t (E_n x)^*(s)\, ds \leq \int_0^t x^*(s)\, ds, \qquad 0 \leq t \leq 1.$$

From Definition 3, we see that $E_n x \in X$ and $\|E_n x\|_X \leq \|x\|_X$ whenever $x \in X$. Thus, $E_n \in \mathscr{L}(X)$ and $\|E_n\|_{\mathscr{L}(X)} \leq 1$, as desired. In fact, $\|E_n\|_{\mathscr{L}(X)} = 1$ for all n, since $E_n x = x$ for every $\mathscr{F}_n$-measurable x. ∎

Remarks. (i) We cannot remove the hypothesis $\sup_n \|E_n\|_{\mathscr{L}(X)} < \infty$ in Theorem 1, as Example 2 shows in Section 4.

(ii) There exists a Banach function space X which is not r.i. and a filtration $(\mathscr{F}_n)_{n \geq 1}$ such that $\sup_n \|E_n\|_{\mathscr{L}(X)} < \infty$; see Section 3.

In the proof of the following lemma, we will use the criterion for uniform integrability of random variables due to la Vallée-Poussin (see [3, p. 38]). Recall that a family $\mathscr{H} \subset L^1$ is uniformly integrable if and only if there exists a nonnegative convex increasing function Φ such that $\Phi(t)/t \to \infty$ as $t \to \infty$ and

$$\sup_{x \in \mathscr{H}} E[\Phi(|x|)] =: \alpha < \infty. \tag{5}$$

Note that we may assume $\alpha \leq 1$ by taking $\alpha^{-1}\Phi$ instead of Φ, if necessary. Hence, Eq. (5) can be replaced by

$$\sup_{x \in \mathscr{H}} \|x\|_\Phi \leq 1,$$

where $\|\cdot\|_\Phi$ denotes the Luxemburg norm ([1, p. 268]).

LEMMA 1. *Let X be a Banach function space, $f = (f_n)_{n \geq 1}$ a martingale, and $(w_n)_{n \geq 1}$ a weight sequence. If $(\sigma_n(f))_{n \geq 1}$ converges in X, then $f = (f_n)$ is uniformly integrable and $f_\infty = \lim_{n \to \infty} f_n \in X$.*

Proof. Assume that $(\sigma_n(f))$ converges in X. As $\sup_n \|\sigma_n(f)\|_X < \infty$, we have $\sup_n \|\sigma_n(f)\|_1 < \infty$ by (a) of Definition 1. We may apply Corollary 1 with $X = L^1$ (or Theorem 1 in [7]) to obtain $\sup_n \|f_n\|_1 < \infty$. Thus $f = (f_n)$ converges a.s., and hence $(\sigma_n(f))$ converges a.s. On the other hand, again by (a) of Definition 1, $(\sigma_n(f))$ converges also in L^1. It follows that $(\sigma_n(f))$ is uniformly integrable (see, e.g., [3, p. 36]). Then there exists a nonnegative increasing convex function Φ satisfying $\sup_n \|\sigma_n(f)\|_\Phi \leq 1$ and $\lim_{t \to \infty} \Phi(t)/t = \infty$. Since the Orlicz space L^Φ is u.r.i., Corollary 1 shows that $\sup_n \|f_n\|_\Phi \leq 1$, which implies the uniform integrability of $f = (f_n)$.

Now let $f_\infty = \lim_n f_n$ a.s. Since $\inf_{k \geq n} |\sigma_k(f)| \uparrow f_\infty$ as $n \to \infty$ and $\sup_n \|\sigma_n(f)\|_X < \infty$, we have $f_\infty \in X$ and $\|f_\infty\|_X \leq \sup_n \|\sigma_n(f)\|_X$ by (c) of Definition 1. This completes the proof. ∎

The following lemma shows that if a weight sequence $(w_n)_{n \geq 1}$ increases very rapidly, then $f = (f_n)_{n \geq 1}$ and $(\sigma_n(f))_{n \geq 1}$ converge simultaneously in any Banach function space X.

LEMMA 2. *Let X be an arbitrary Banach function space, $f = (f_n)_{n \geq 1}$ a martingale, and $(w_n)_{n \geq 1}$ a weight sequence such that*

$$\varliminf_{n \to \infty} \frac{w_n}{W_n} > 0. \tag{6}$$

Then $f = (f_n)$ converges in X if and only if $(\sigma_n(f))$ converges in X.

Proof. Assume that $(\sigma_n(f))$ converges in X. Then $f = (f_n)$ is uniformly integrable by Lemma 1, and $\sigma_n(f) \to f_\infty$ in X, where $f_\infty = \lim_n f_n$ a.s. For every $n \geq 2$, we have

$$\sigma_n(f) - f_\infty = \frac{w_n}{W_n}(f_n - f_\infty) + \frac{W_{n-1}}{W_n}(\sigma_{n-1}(f) - f_\infty),$$

and therefore,

$$\begin{aligned}\frac{w_n}{W_n}\|f_n - f_\infty\|_X &\leq \|\sigma_n(f) - f_\infty\|_X + \frac{W_{n-1}}{W_n}\|\sigma_{n-1}(f) - f_\infty\|_X \\ &\leq \|\sigma_n(f) - f_\infty\|_X + \|\sigma_{n-1}(f) - f_\infty\|_X.\end{aligned}$$

Since the right-hand side tends to zero, Eq. (6) shows that $\|f_n - f_\infty\|_X \to 0$ as $n \to \infty$.

Conversely, assume that $f = (f_n)$ converges in X. Then $f = (f_n)$ is uniformly integrable, since it converges a.s. and in L^1. Since $\|f_k - f_\infty\|_X \to 0$ as $k \to \infty$, the inequality

$$\|\sigma_n(f) - f_\infty\|_X \leq \frac{1}{W_n}\sum_{k=1}^{n} w_k\|f_k - f_\infty\|_X$$

yields that $\sigma_n(f) \to f_\infty$ in X. The lemma is established. ∎

We are now in a position to prove our main theorem.

THEOREM 2. *Let X be a Banach function space, $f = (f_n, \mathscr{F}_n)_{n\geq 1}$ a martingale, and $(w_n)_{n\geq 1}$ a weight sequence. Suppose that $E_n = E[\cdot|\mathscr{F}_n] \in \mathscr{L}(X)$ for all $n \geq 1$ and*

$$\sup_n \|E_n\|_{\mathscr{L}(X)} =: C < \infty. \tag{7}$$

Then $f = (f_n)$ converges in X if and only if $(\sigma_n(f))$ converges in X.

Proof. It suffices to show that if $(\sigma_n(f))$ converges in X, then $f = (f_n)$ converges in X. Suppose first that $\sup_{n\geq 1} W_{n+1}/W_n =: K < \infty$. According to Lemma 1, $f = (f_n)$ is uniformly integrable. We have by Eq. (4) that

$$E[\sigma_{n+1}(f) - f_\infty|\mathscr{F}_n] = \frac{W_n}{W_{n+1}}(\sigma_n(f) - f_n),$$

and therefore by Eq. (7) that

$$\begin{aligned}\frac{W_n}{W_{n+1}}\|\sigma_n(f) - f_n\|_X &\leq \|E[\sigma_{n+1}(f) - f_\infty|\mathscr{F}_n]\|_X \\ &\leq C\|\sigma_{n+1}(f) - f_\infty\|_X\end{aligned}$$

for every $n \geq 1$. Then it follows that

$$\begin{aligned}\|f_n - f_\infty\|_X &\leq \frac{W_{n+1}}{W_n}\left(\frac{W_n}{W_{n+1}}\|\sigma_n(f) - f_n\|_X + \frac{W_n}{W_{n+1}}\|\sigma_n(f) - f_\infty\|_X\right)\\ &\leq K\big(C\|\sigma_{n+1}(f) - f_\infty\|_X + \|\sigma_n(f) - f_\infty\|_X\big).\end{aligned}$$

Since the right-hand side tends to zero, we see that $f_n \to f_\infty$ in X.

Now we remove the additional assumption $\sup_n W_{n+1}/W_n < \infty$. Assume $\sup_n W_{n+1}/W_n = \infty$ or equivalently, $\sup_n w_{n+1}/W_n = \infty$, and choose a subsequence $(w_{n_j})_{j\geq 1}$ so that

$$\lim_{j\to\infty} \frac{w_{n_j}}{W_{n_j - 1}} = \infty. \tag{8}$$

For each integer $j \geq 1$, set $\overline{w}_j = w_{n_j}$ and $\overline{W}_k = \sum_{j=1}^k \overline{w}_j$. It is clear from Eq. (8) that $\overline{w}_j \to \infty$ as $j \to \infty$ and hence that $\overline{W}_k \to \infty$ as $k \to \infty$. Furthermore, since

$$\overline{W}_k = w_{n_1} + \cdots + w_{n_{k-1}} + w_{n_k} \leq W_{n_k - 1} + \overline{w}_k,$$

we have

$$1 + \frac{W_{n_k-1}}{w_{n_k}} = \frac{W_{n_k}}{\overline{w}_k} \geq \frac{W_{n_k}}{\overline{W}_k} \geq 1 \geq \frac{\overline{w}_k}{\overline{W}_k} \geq \frac{\overline{w}_k}{W_{n_k-1} + \overline{w}_k} \geq 1 - \frac{W_{n_k-1}}{w_{n_k}}.$$

Thus Eq. (8) gives that

$$\lim_{k\to\infty} \frac{W_{n_k}}{\overline{W}_k} = \lim_{k\to\infty} \frac{\overline{w}_k}{\overline{W}_k} = 1. \tag{9}$$

Now for each $k \geq 1$, put

$$\overline{\sigma}_k(f) = \frac{1}{\overline{W}_k} \sum_{j=1}^{k} \overline{w}_j f_{n_j}.$$

In other words, $(\overline{\sigma}_k(f))_{k\geq 1}$ is the weighted average of the martingale $\bar{f} = (f_{n_k}, \mathscr{F}_{n_k})_{k\geq 1}$ with respect to $(\overline{w}_k)_{k\geq 1}$. We show that $(\overline{\sigma}_k(f))$ converges to f_∞ in X. Observe that

$$\sigma_{n_k}(f) - f_\infty = \frac{\overline{W}_k}{W_{n_k}}\big(\overline{\sigma}_k(f) - f_\infty\big) + \frac{1}{W_{n_k}} \sideset{}{'}\sum_{m=1}^{n_k} w_m(f_m - f_\infty),$$

where Σ' denotes the sum over $m \le n_k$ such that $m \ne n_j$ for any j. Therefore, since $\|f_m\|_X \le C\|f_\infty\|$ by Eq. (7), we have that

$$\begin{aligned}
&\frac{\overline{W}_k}{W_{n_k}} \| \overline{\sigma}_k(f) - f_\infty \|_X \\
&\quad \le \left\| \sigma_{n_k}(f) - f_\infty \right\|_X + \frac{1}{W_{n_k}} \sum_{m=1}^{n_k}{}' w_m \|f_m - f_\infty\|_X \\
&\quad \le \left\| \sigma_{n_k}(f) - f_\infty \right\|_X + \frac{C+1}{W_{n_k}} \|f_\infty\|_X \sum_{m=1}^{n_k}{}' w_m \\
&\quad = \left\| \sigma_{n_k}(f) - f_\infty \right\|_X + (C+1)\|f_\infty\|_X \left(1 - \frac{\overline{W}_k}{W_{n_k}} \right).
\end{aligned}$$

From Eq. (9) and the above inequalities, it follows that $\|\overline{\sigma}_k(f) - f_\infty\|_X \to 0$, as was to be shown.

Now we prove that $f = (f_n)$ converges in X. Since $(\overline{\sigma}_k(f))$ converges in X and $\overline{W}_k$ satisfies Eq. (9), Lemma 2 shows that $\bar{f} = (f_{n_k})_{k \ge 1}$ converges in X. If $m, n \ge n_k$, then

$$\begin{aligned}
\|f_n - f_m\|_X &\le \left\| E\left[f_\infty - f_{n_k} | \mathscr{F}_n \right] \right\|_X + \left\| E\left[f_\infty - f_{n_k} | \mathscr{F}_m \right] \right\|_X \\
&\le 2C\|f_\infty - f_{n_k}\|_X \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

This means that $f = (f_n)$ is a Cauchy sequence in X and hence it converges in X. The theorem is established. ▮

Remark. In Theorem 2, the hypothesis $\sup_n \|E_n\|_{\mathscr{L}(X)} < \infty$ is essential; see Example 1 in Section 4.

The equivalence between (a) and (b) of the following theorem was given in [8], and in [9] in more general form.

THEOREM 3. *Let X be a Banach function space, $f = (f_n, \mathscr{F}_n)_{n \ge 1}$ a martingale, and $(w_n)_{n \ge 1}$ a weight sequence. If X has absolutely continuous norm and E_n satisfy Eq.* (7), *then the following conditions are equivalent*:

(a) $f = (f_n)$ *is uniformly integrable and* $f_\infty \in X$;

(b) $f = (f_n)$ *converges in* X;

(c) $(\sigma_n(f))$ *converges in* X.

Proof. In view of Lemma 1, it is clear that (b) implies (c), and (c) implies (a). Therefore, it suffices to show that (a) implies (b). To this end, we may assume that $\|1\|_X = 1$. First suppose that $f_\infty \in L^\infty$. For any $\varepsilon > 0$, we have

$$\begin{aligned}\|f_n - f_\infty\|_X &\le \left\|(f_n - f_\infty)1_{\{|f_n - f_\infty| \le \varepsilon\}}\right\|_X + \left\|(f_n - f_\infty)1_{\{|f_n - f_\infty| > \varepsilon\}}\right\|_X \\ &\le \varepsilon + 2\|f_\infty\|_\infty \|1_{\{|f_n - f_\infty| > \varepsilon\}}\|_X .\end{aligned}$$

Since $f_n \to f_\infty$ a.s. and X has absolutely continuous norm, the last term on the right-hand side tends to zero as $n \to \infty$. Letting $\varepsilon \to 0$, we obtain $\lim_n \|f_n - f_\infty\|_X = 0$.

Now assume that $f_\infty \in X$. Given an $\varepsilon > 0$, choose a random variable $g_\infty \in L^\infty$ so that $\|f_\infty - g_\infty\|_X < \varepsilon$. This is possible, because L^∞ is dense in X as mentioned in Section 1. Using Eq. (7), we have

$$\begin{aligned}\|f_n - f_\infty\|_X &\le \|E[f_\infty - g_\infty | \mathscr{F}_n]\|_X + \|g_n - g_\infty\|_X + \|g_\infty - f_\infty\|_X \\ &\le (C+1)\varepsilon + \|g_n - g_\infty\|_X .\end{aligned}$$

By what we have proved above, the last term on the right-hand side tends to zero as $n \to \infty$. Letting $\varepsilon \to 0$, we have $\lim_n \|f_n - f_\infty\|_X = 0$. This completes the proof. ∎

We have shown in the proof of Corollary 1 that if X is u.r.i., then $E_n \in \mathscr{L}(X)$ and $\sup_n \|E_n\|_{\mathscr{L}(X)} = 1$ for any filtration $(\mathscr{F}_n)$. Hence we have the following.

COROLLARY 2. *Suppose that X is a u.r.i. space* (*or r.i. space if* Ω *contains no atom*) *and $f = (f_n)_{n \ge 1}$ is a martingale. Then* (b) *and* (c) *of Theorem* 3 *are equivalent. Furthermore, if X has absolutely continuous norm, then* (a), (b), *and* (c) *of Theorem* 3 *are equivalent.*

Remark. Suppose that Ω contains no atom and X is a r.i. space. Under these conditions, we proved in [9] that, if (a) of Theorem 3 implies (b) of Theorem 3, then X has absolutely continuous norm. In Theorem 2, however, it is unnecessary to assume that X has absolutely continuous norm. Let $\varphi \ge 0$ be a nondecreasing concave function on the interval $[0, 1]$, and $M(\varphi)$ be the *Lorentz space* consisting of all random variables x such that

$$\|x\|_{M(\varphi)} = \sup_{0 < t \le 1} \frac{\varphi(t)}{t} \int_0^t x^*(s)\, ds < \infty.$$

In general, $M(\varphi)$ does not have absolutely continuous norm, while $M(\varphi)$ is a r.i. space (see [1, pp. 67–69]). Hence, if $X = M(\varphi)$, then (b) and (c) of Theorem 3 are equivalent by Theorem 2, but (a) and (b) are not equivalent. Note that the same is true for $X = L^\infty$, since $M(\varphi) = L^\infty$ when $\varphi \equiv 1$.

3. CONVERGENCE IN WEIGHTED L^p-SPACES

We shall investigate the convergence of a martingale and its weighted average in weighted L^p-spaces.

Izumisawa and Kazamaki [6] studied some martingale inequalities in weighted L^p-spaces and introduced the condition A_p, the probabilistic analog of the condition A_p in classical analysis. In this section, we shall see that the condition A_p is suitable for describing our result.

Let $z = (z_n, \mathscr{F}_n)_{n \geq 1}$ be a uniformly integrable martingale such that $z_\infty > 0$ a.s. and $E[z_\infty] = 1$. A martingale $z = (z_n)$ satisfying these conditions is called a *weight martingale*. Let $1 \leq p < \infty$ and suppose that

$$z_\infty^{-1/(p-1)} \in L^1 \quad \text{or} \quad z_\infty^{-1} \in L^\infty, \tag{10}$$

according as $p > 1$ or $p = 1$. We denote by $L^p(z)$ the space of all random variables x such that

$$\|x\|_{p,z} := E\left[|x|^p z_\infty\right]^{1/p} < \infty.$$

Then $(L^p(z), \|\cdot\|_{p,z})$ is a Banach function space. In fact, (b) and (c) of Definition 1 are obvious, and (a) of Definition 1 immediately follows from Eq. (10) and Hölder's inequality.

The weight martingale $z = (z_n, \mathscr{F}_n)$ is said to satisfy A_p (with respect to $(\mathscr{F}_n)$) if

$$\sup_n E\left[\left(\frac{z_n}{z_\infty}\right)^{1/(p-1)} \Bigg| \mathscr{F}_n\right]^{p-1} \leq K \quad \text{a.s.} \qquad \text{or} \qquad \sup_n \frac{z_n}{z_\infty} \leq K \quad \text{a.s.}, \tag{A_p}$$

according as $p > 1$ or $p = 1$, where K is a positive constant.

The following lemma is due to Doléans-Dade and Meyer [5]. We shall present a simplification of their proof.

LEMMA 3. *Let $1 \leq p < \infty$ and $z = (z_n, \mathscr{F}_n)_{n \geq 1}$ be a weight martingale satisfying Eq.* (10). *Then the following conditions are equivalent*:

(a) $E_n \in \mathscr{L}(L^p(z))$ *for all $n \geq 1$ and* $\sup_n \|E_n\|_{\mathscr{L}(L^p(z))} \leq K^{1/p}$;

(b) $z = (z_n, \mathscr{F}_n)$ *satisfies Eq.* (A_p).

Proof. Assume (b) and let $x \in L^p(z)$. By Hölder's inequality, we have

$$\begin{aligned} \left|E[x \mid \mathscr{F}_n]\right|^p z_n &\leq E\left[z_\infty^{1/p} z_\infty^{-1/p} |x| \mid \mathscr{F}_n\right]^p z_n \\ &\leq E\left[|x|^p z_\infty \mid \mathscr{F}_n\right] E\left[(z_n/z_\infty)^{1/(p-1)} \mid \mathscr{F}_n\right]^{p-1}, \end{aligned}$$

provided $1 < p < \infty$. Hence, Eq. (A_p) shows that

$$|E[x\,|\,\mathscr{F}_n]|^p z_n \le KE\left[|x|^p z_\infty\,|\,\mathscr{F}_n\right].$$

Obviously, this inequality is also valid when $p = 1$. Thus we have

$$E\left[|E[x\,|\,\mathscr{F}_n]|^p z_\infty\right] = E\left[|E[x\,|\,\mathscr{F}_n]|^p z_n\right] \le KE\left[|x|^p z_\infty\right],$$

which implies (a).

Conversely, suppose that (a) is true. Then for any nonnegative random variable x, we have

$$E\left[E[x\,|\,\mathscr{F}_n]^p z_n\right] \le KE[x^p z_\infty]. \tag{11}$$

We first assume that $1 < p < \infty$. Let $\varepsilon > 0$ and set $x = z_\infty^{-[1/(p-1)]} 1_{\{z_\infty > \varepsilon\} \cap A}$ in Eq. (11), where $A \in \mathscr{F}_n$. It follows that

$$E\left[E\left[z_\infty^{-[1/(p-1)]} 1_{\{z_\infty > \varepsilon\}}\,|\,\mathscr{F}_n\right]^p z_n 1_A\right] \le KE\left[z_\infty^{-[1/(p-1)]} 1_{\{z_\infty > \varepsilon\}} 1_A\right].$$

Since $A \in \mathscr{F}_n$ is arbitrary, we have

$$E\left[z_\infty^{-[1/(p-1)]} 1_{\{z_\infty > \varepsilon\}}\,|\,\mathscr{F}_n\right]^p z_n \le KE\left[z_\infty^{-[1/(p-1)]} 1_{\{z_\infty > \varepsilon\}}\,|\,\mathscr{F}_n\right] \quad \text{a.s.},$$

or equivalently,

$$E\left[(z_n/z_\infty)^{1/(p-1)} 1_{\{z_\infty > \varepsilon\}}\,|\,\mathscr{F}_n\right]^{p-1} \le K \quad \text{a.s.}$$

Letting $\varepsilon \to 0$, we see that $z = (z_n)$ satisfies Eq. (A_p).

Now assume $p = 1$. Then Eq. (11) can be rewritten as $E[xz_n] \le KE[xz_\infty]$. Setting $x = z_\infty^{-1} 1_A$ for an arbitrary $A \in \mathscr{F}$, we have

$$E\left[(z_n/z_\infty) 1_A\right] \le KP(A).$$

This implies that $z_n/z_\infty \le K$ a.s. and the lemma is established. ∎

Notice that, if $1 \le p < \infty$, then $L^p(z)$ has absolutely continuous norm. Therefore, from Theorem 3 and Lemma 3, we obtain the following.

THEOREM 4. *Let $z = (z_n, \mathscr{F}_n)_{n \ge 1}$ be a weight martingale satisfying Eq.* (10), *$f = (f_n, \mathscr{F}_n)_{n \ge 1}$ a martingale, and $(w_n)_{n \ge 1}$ a weight sequence. If $z = (z_n, \mathscr{F}_n)$ satisfies, Eq. (A_p), then the following conditions are equivalent*:

(a) *$f = (f_n)$ is uniformly integrable and $f_\infty \in L^p(z)$;*

(b) *$f = (f_n)$ converges in $L^p(z)$;*

(c) *$(\sigma_n(f))$ converges in $L^p(z)$.*

4. EXAMPLES

In this section, we shall give examples which show that we cannot remove the hypothesis $\sup_n \|E_n\|_{\mathscr{L}(X)} < \infty$ in Theorems 1 and 2. We shall use the following two elementary lemmas.

LEMMA 4. *Let $(w_n)_{n \geq 1}$ be a weight sequence. If $\underline{\lim}_n w_n/W_n = 0$, then there exists a subsequence $(w_{n_k})_{k \geq 1}$ such that*

$$\lim_{k \to \infty} \frac{1}{W_{n_k}} (w_{n_1} + w_{n_2} + \cdots + w_{n_k}) = 0. \tag{12}$$

Proof. We can find a subsequence (w_{n_k}) such that $\sum_{k=1}^{\infty} w_{n_k}/W_{n_k} < \infty$ by hypothesis. Then (w_{n_k}) satisfies Eq. (12). Indeed, we have for every $k > m$,

$$\begin{aligned} \frac{1}{W_{n_k}} \sum_{j=1}^{k} w_{n_j} &\leq \frac{1}{W_{n_k}} \sum_{j=1}^{m} w_{n_j} + \sum_{j=m+1}^{k} \frac{w_{n_j}}{W_{n_j}} \\ &\leq \frac{1}{W_{n_k}} \sum_{j=1}^{m} w_{n_j} + \sum_{j=m+1}^{\infty} \frac{w_{n_j}}{W_{n_j}}. \end{aligned}$$

This shows that, for each $m \geq 1$,

$$\overline{\lim_{k \to \infty}} \frac{1}{W_{n_k}} (w_{n_1} + \cdots + w_{n_k}) \leq \sum_{j=m+1}^{\infty} \frac{w_{n_j}}{W_{n_j}}.$$

Letting $m \to \infty$, we obtain Eq. (12). ∎

LEMMA 5. *Let $(w_n)_{n \geq 1}$ be a weight sequence, $(w_{n_k})_{k \geq 1}$ be a subsequence satisfying Eq.* (12), *and $\mathscr{N}$ denote the set of n_k, $k \geq 1$. If $(a_n)_{n \geq 1}$ is a bounded sequence of nonnegative numbers such that*

$$\lim_{\substack{n \to \infty \\ n \notin \mathscr{N}}} a_n = 0,$$

then $(1/W_n)(w_1 a_1 + w_2 a_2 + \cdots + w_n a_n) \to 0$ as $n \to \infty$.

Proof. Given an $\varepsilon > 0$, choose an integer N such that $0 \leq a_j \leq \varepsilon$ whenever $j > N$ and $j \notin \mathscr{N}$. For each $n > N$, put $p(n) = \max\{n_k \colon n_k \leq n\}$. Then we have

$$\begin{aligned} \frac{1}{W_n}(w_1 a_1 + \cdots + w_n a_n) &= \frac{1}{W_n} \left(\sum_{\substack{j=N+1 \\ j \notin \mathscr{N}}}^{n} w_j a_j + \sum_{\substack{j=N+1 \\ j \in \mathscr{N}}}^{n} w_j a_j + \sum_{j=1}^{N} w_j a_j \right) \\ &\leq \varepsilon + \frac{K}{W_{p(n)}} \sum_{\substack{j=1 \\ j \in \mathscr{N}}}^{p(n)} w_j + \frac{1}{W_n} \sum_{j=1}^{N} w_j a_j, \end{aligned}$$

where K denotes an upper bound of (a_n). Since $p(n) \to \infty$ as $n \to \infty$, we have by Eq. (12) that

$$\overline{\lim_{n\to\infty}} \frac{1}{W_n}(w_1 a_1 + \cdots + w_n a_n) \le \varepsilon,$$

which completes the proof. ∎

EXAMPLE 1. Let $\Omega = [0,1]$ be the probability space with Lebesgue measure P and the σ-field of Lebesgue measurable sets $\mathcal{F}$. We denote by X the Banach space of random variables x on Ω such that

$$\|x\|_X := \left(\int_0^1 |x(t)|^p \, d\sqrt{t} \right)^{1/p} < \infty,$$

where $1 \le p < \infty$ is arbitrarily fixed. If $1 < p < \infty$ and $x \in X$, then Hölder's inequality shows that

$$\begin{aligned} \|x\|_1 &\le \left(\int_0^1 |x(t)|^p \frac{dt}{2\sqrt{t}} \right)^{1/p} \left(\int_0^1 (2\sqrt{t})^{p'/p} \, dt \right)^{1/p'} \\ &= 2\left(\frac{p-1}{2p-1} \right)^{1/p'} \|x\|_X, \end{aligned}$$

where $p' = p/(p-1)$. Thus we see that $X \hookrightarrow L^1$. Evidently, we have the same embedding also for $p = 1$. On the other hand, we have $L^\infty \hookrightarrow X$; hence X satisfies (a) of Definition 1. Since X satisfies (b) and (c) of Definition 1, $(X, \|\cdot\|_X)$ is a Banach function space.

Let f_∞ be the random variable on Ω given by

$$f_\infty(t) = \begin{cases} (1-t)^{-1/2p}, & \text{if } 1/2 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to see that $f_\infty \in X$ and $\|f_\infty\|_X = (\pi/4)^{1/p}$.

For integers $m, n \ge 2$, we put

$$A_m = \left[0, \tfrac{1}{m}\right], \qquad B_n = \left[1 - \tfrac{1}{n}, 1\right], \qquad C_{m,n} = A_m \cup B_n.$$

Let $\mathcal{F}_{m,n}$ denote the σ-field generated by $C_{m,n}$ and the measurable subsets of $\Omega \setminus C_{m,n}$; $C_{m,n}$ is a single atom with respect to $\mathcal{F}_{m,n}$ and P. Put

$$f_{m,n} = E[f_\infty \,|\, \mathcal{F}_{m,n}], \qquad m, n \ge 2.$$

It follows that

$$\begin{aligned} f_{m,n} &= f_\infty 1_{\Omega \setminus C_{m,n}} + \frac{1_{C_{m,n}}}{P(C_{m,n})} \int_{C_{m,n}} f_\infty \, dP \\ &= f_\infty 1_{\Omega \setminus C_{m,n}} + \frac{2p}{2p-1} \cdot \frac{mn^{1/2p}}{m+n} 1_{C_{m,n}}. \end{aligned}$$

Note that

$$\|1_{C_{m,n}}\|_X = \left\{m^{-1/2} + 1 - \left(1 - n^{-1}\right)^{1/2}\right\}^{1/p}$$
$$\leq \left(m^{-1/2} + n^{-1/2}\right)^{1/p} \leq m^{-1/2p} + n^{-1/2p}.$$

Using these facts, we have the following inequalities:

$$\begin{aligned}\|f_{m,n} - f_\infty\|_X &= \left\|(f_{m,n} - f_\infty)1_{C_{m,n}}\right\|_X \\ &\leq \|f_{m,n}1_{C_{m,n}}\|_X + \|f_\infty 1_{B_n}\|_X \\ &= \frac{2p}{2p-1} \cdot \frac{mn^{1/2p}}{m+n}\left(m^{-1/2p} + n^{-1/2p}\right) + \|f_\infty 1_{B_n}\|_X \\ &= \frac{2p}{2p-1} \cdot \frac{m^{1-1/2p}n^{1/2p} + m}{m+n} + \|f_\infty 1_{B_n}\|_X \\ &\leq \frac{4p}{2p-1} + \left(\frac{\pi}{4}\right)^{1/p}.\end{aligned}$$

Since $\|f_\infty 1_{B_n}\|_X \to 0$ by the dominated convergence theorem, we see that

(i) $\lim_{n\to\infty} \|f_{m,n} - f_\infty\|_X = 0$ for each fixed $m \geq 2$;

(ii) $\|f_{m,n} - f_\infty\|_X \leq K := 4p(2p-1)^{-1} + (\pi/4)^{1/p}$ for all $m, n \geq 2$.

On the other hand, since $f_{n,n} = p(2p-1)^{-1}n^{1/2p}$ on A_n, we have

$$\|f_{n,n} - f_\infty\|_X \geq \|f_{n,n}1_{A_n}\|_X = \frac{p}{2p-1}n^{1/2p}\|1_{A_n}\|_X = \frac{p}{2p-1}$$

for every $n \geq 2$.

Now assume that $\underline{\lim}_n w_n/W_n = 0$. We shall construct a uniformly integrable martingale $f = (f_n, \mathscr{F}_n)_{n\geq 1}$ such that $\|\sigma_n(f) - f_\infty\|_X \to 0$ and $\|f_n - f_\infty\|_X \nrightarrow 0$.

According to Lemma 4, there is a subsequence $(w_{n_k})_{k\geq 1}$ satisfying Eq. (12). Let $m_0 = 2$ and choose an integer $N_0 \geq 2$ so that $\|f_{m_0,n} - f_\infty\|_X \leq 1$ whenever $n > N_0$. This is possible by (i) above. Then we define f_j and $\mathscr{F}_j$, $j = 1, 2, \dots, n_1$, as follows: set $m_1 = N_0 + n_1$ and

$$f_j = \begin{cases} f_{m_0, N_0+j}, & j = 1, 2, \dots, n_1 - 1, \\ f_{m_1, m_1}, & j = n_1, \end{cases}$$

$$\mathscr{F}_j = \begin{cases} \mathscr{F}_{m_0, N_0+j}, & j = 1, 2, \dots, n_1 - 1, \\ \mathscr{F}_{m_1, m_1}, & j = n_1. \end{cases}$$

It follows that $\mathscr{F}_j \subset \mathscr{F}_{j+1}$ and $\|f_j - f_\infty\|_X \le 1$ for $j = 1, 2, \ldots, n_1 - 1$, and that $p/(2p-1) \le \|f_{n_1} - f_\infty\|_X \le K$, where K is the constant given in (ii).

Now choose $N_1 \ge m_1$ so that $\|f_{m_1, n} - f_\infty\|_X \le 1/2$ whenever $n > N_1$, and then define f_{n_1+j} and $\mathscr{F}_{n_1+j}$, $j = 1, 2, \ldots, n_2 - n_1$, as follows: set $m_2 = N_1 + n_2 - n_1$ and

$$f_{n_1+j} = \begin{cases} f_{m_1, N_1+j}, & j = 1, 2, \ldots, n_2 - n_1 - 1, \\ f_{m_2, m_2}, & j = n_2 - n_1, \end{cases}$$

$$\mathscr{F}_{n_1+j} = \begin{cases} \mathscr{F}_{m_1, N_1+j}, & j = 1, 2, \ldots, n_2 - n_1 - 1, \\ \mathscr{F}_{m_2, m_2}, & j = n_2 - n_1. \end{cases}$$

Thus we obtain an increasing family $(\mathscr{F}_j)_{j=1}^{n_2}$ of sub-σ-fields of $\mathscr{F}$ and a martingale $(f_j, \mathscr{F}_j)_{j=1}^{n_2}$ such that

$$\|f_j - f_\infty\|_X \le 1, \qquad 1 \le j < n_1,$$

$$\|f_j - f_\infty\|_X \le 1/2, \qquad n_1 < j < n_2,$$

and

$$\frac{p}{2p-1} \le \|f_{n_k} - f_\infty\|_X \le K, \qquad k = 1, 2.$$

Define inductively f_j and $\mathscr{F}_j$ in the same manner as above. Then we obtain a martingale $f = (f_j, \mathscr{F}_j)_{j \ge 1}$ such that

$$\|f_j - f_\infty\|_X \le 1/k, \qquad n_{k-1} < j < n_k, \quad k \ge 1, \tag{13}$$

$$\frac{p}{2p-1} \le \|f_{n_k} - f_\infty\|_X \le K, \qquad k \ge 1, \tag{14}$$

where $n_0 = 0$. Let $\mathscr{N}$ denote the set of $n_1, n_2, \ldots$. Then Eq. (13) gives that

$$\lim_{\substack{j \to \infty \\ j \notin \mathscr{N}}} \|f_j - f_\infty\|_X = 0.$$

In view of Lemma 5, we see that

$$\|\sigma_n(f) - f_\infty\|_X \le \frac{1}{W_n} \sum_{j=1}^{n} w_j \|f_j - f_\infty\|_X \to 0 \quad \text{as } n \to \infty,$$

while $\|f_n - f_\infty\|_X \nrightarrow 0$ by Eq. (14).

Now note that every $E_{m,n} = E[\cdot|\mathscr{F}_{m,n}]$ is a bounded linear operator on X into itself. This follows from the inequalities

$$\|E[x|\mathscr{F}_{m,n}]\|_X \leq \left(\frac{1}{P(C_{m,n})}\int_{C_{m,n}}|x|\,dP\right)\|1_{C_{m,n}}\|_X + \|x1_{\Omega\setminus C_{m,n}}\|_X$$
$$\leq c\|x\|_1 + \|x\|_X \leq c'\|x\|_X,$$

where c and c' are positive constants depending on m and n, but independent of $x \in X$. In particular, $E_n = E[\cdot|\mathscr{F}_n] \in \mathscr{L}(X)$ for every $n \geq 1$. But Theorem 2 shows that $\sup_n \|E_n\|_{\mathscr{L}(X)} = \infty$. Indeed, we have

$$\frac{\left\|E\left[1_{B_{n_k}}|\mathscr{F}_{n_k}\right]\right\|_X}{\|1_{B_{n_k}}\|_X} = \left(\sqrt{n_k} + \sqrt{n_k - 1} + 1\right)^{1/p} \to \infty \quad \text{as } k \to \infty.$$

EXAMPLE 2. Let $(\Omega, \mathscr{F}, P)$ be as in Example 1. For each random variable x on Ω, we put

$$\|x\|_X = \int_0^{1/2} |x(t)|\,d\sqrt{t} + \int_{1/2}^1 |x(t)|\,dt.$$

Let X denote the set of all x such that $\|x\|_X < \infty$. Then it is easy to see that $(X, \|\cdot\|_X)$ is a Banach function space.

Let $\mathscr{F}_{m,n}$ be the σ-field defined in Example 1, and set

$$f_{m,n} = \frac{mn}{m+n}1_{C_{m,n}}, \qquad m, n \geq 2.$$

Then each $f_{m,n}$ is $\mathscr{F}_{m,n}$-measurable and

$$E\left[f_{m,n}|\mathscr{F}_{m',n'}\right] = f_{m',n'} \quad \text{if } 2 \leq m' \leq m \text{ and } 2 \leq n' \leq n. \tag{15}$$

For each $n \geq 2$, we have

$$\|f_{n^2,n}\|_X = \frac{2n}{n+1} \leq 2, \qquad \|f_{n,n}\|_X = \frac{1}{2}(1 + \sqrt{n}). \tag{16}$$

We construct a martingale $f = (f_n)_{n\geq 1}$ such that $\sup_n \|\sigma_n(f)\|_X < \infty$ for some weight sequence (w_n), and that $\sup_n \|f_n\|_X = \infty$. For each $j \geq 1$, we set

$$m_j = \begin{cases} 2^{2^{k-1}}, & \text{if } j = 2k-1, \\ 2^{2^k}, & \text{if } j = 2k, \end{cases}$$

and

$$n_j = \begin{cases} 2^{2^k}, & \text{if } j = 2k-1, \\ 2^{2^k}, & \text{if } j = 2k. \end{cases}$$

It is clear that both $(m_j)_{j\geq 1}$ and $(n_j)_{j\geq 1}$ are nondecreasing in j, and that

$$m_j^2 = n_j \quad \text{if } j \text{ is odd;} \qquad m_j = n_j \quad \text{if } j \text{ is even.} \tag{17}$$

Put $f_j = f_{m_j, n_j}$ and $\mathscr{F}_j = \mathscr{F}_{m_j, n_j}$, $j = 1, 2, \dots$. Then Eq. (15) shows that $f = (f_j, \mathscr{F}_j)_{j\geq 1}$ is a martingale. From Eqs. (16) and (17), we see that $\|f_j\|_X = \frac{1}{2}(1 + \sqrt{n_j})$ if j is even, and hence $\sup_j \|f_j\|_X = \infty$. Let $(w_j)_{j\geq 1}$ be the sequence given by

$$w_j = \begin{cases} 1, & \text{if } j \text{ is odd,} \\ \left(1 + \sqrt{n_j}\right)^{-1}, & \text{if } j \text{ is even.} \end{cases}$$

Then, since $W_n = \sum_{j=1}^n w_j \geq n/2$, we have by Eq. (16) that

$$\|\sigma_n(f)\|_X \leq \frac{1}{W_n} \sum_{j=1}^n w_j \|f_j\|_X \leq \frac{2}{n}\left(2 \cdot \frac{n}{2} + \frac{1}{2} \cdot \frac{n}{2}\right) = \frac{5}{2},$$

and thus $\sup_n \|\sigma_n(f)\|_X \leq 5/2$.

We have $\sup_n \|E_n\|_{\mathscr{L}(X)} = \infty$ as in Example 1, and we cannot remove the hypothesis $\sup_n \|E_n\|_{\mathscr{L}(X)} < \infty$ in Theorem 1.

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